MATA22 Individual Assignment

An Exposition about Vectors, Bases
and Coordinates in Three-Dimensional Space

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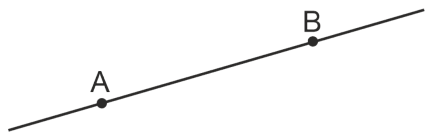
Introduction: An Absurd Interpretation

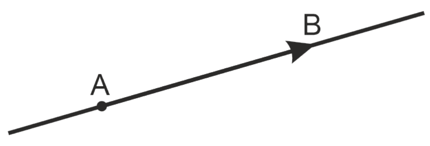
It may sound ridiculously impossible if someone claimed that the whole geometric nature of our wonderful, well-developed, and rather complicated 21st-century-world could be fully understood by spending some time with drawing shapes on paper. You might be under the impression that in order to grasp the true essence of geometry, one must be able to go through massive computations, and probably have a degree in Architecture, Physics, Mathematics, etc. However, what if I told you that every single geometric concept you encounter daily; every building, street, room, and perhaps even the most annoyingly complicated shapes could be professionally interpreted, only if you could work out a clear meaning of a straight *line*?

Lines and Vectors: How our world is shaped

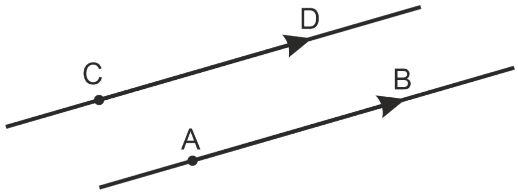
Perhaps, if not the simplest, it happens to be a very trivial challenge to draw a line on paper when even a three-year-old could do it. Now, imagine that you have two points named A and B on a line. As soon as you mark these two points, the part of the line that naturally gets stuck in between A and B is called a *line segment*.



For the sake of our further explanations, let $\overline{AB}$ denote the line segment.
Let us further the imagination by thinking of A and B as start- and endpoint of a traveled path respectively. This path could naturally be illustrated by an arrow, which A acts as its starting point and B would be its tip, illustrated by an arrow head. Therefore, we would have:



Denoted by $\overrightarrow{AB}$, the arrow is called a *Directed Line Segment*, which starts at A and ends at B. Now, assuming that we have two lines instead of one in the first place, we could further our interpretations about the properties of line segments. Assuming that the second line is parallel to our original, it is undoubtedly clear that the discussed aspects about the points A and B also apply to two random points C and D on the second line. That is, there exists a line segment between C and D, denoted by $\overline{CD}$, as well as a directed line segment $\overrightarrow{CD}$.

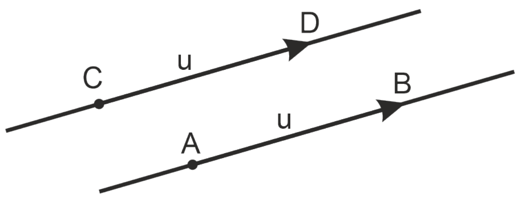


Just as a mathematical terminology, we say that the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are "equivalent" if they have the same direction and magnitude. This relation of equivalence is denoted by:

$$\overrightarrow{AB} \sim \overrightarrow{CD}$$

It is therefore a natural consequence of our interpretations to assume that such properties and equivalences could also apply to three, four, or more parallel lines. The more general question would be: What if we had infinitely many lines? It becomes rather an exhausting process to mark a pair of points on each line and work out the properties. To avoid being caught in an infinite loop, we introduce the concept of *vectors* as a generalization of visualizing equivalent directed line segments.

Let us choose one of the infinitely many equivalent directed line segments. For instance, our original $\overrightarrow{AB}$. The vector u which contains $\overrightarrow{AB}$ is in fact, a set of all directed line segments that are equivalent to $\overrightarrow{AB}$, meaning that both $\overrightarrow{CD}$ and $\overrightarrow{AB}$ as well as all the other equivalent directed line segments belong to the vector u. If we choose $\overrightarrow{AB}$ and investigate the equivalency of all the other directed line segments on u with respect to $\overrightarrow{AB}$, we call $\overrightarrow{AB}$ the *representative* of u.

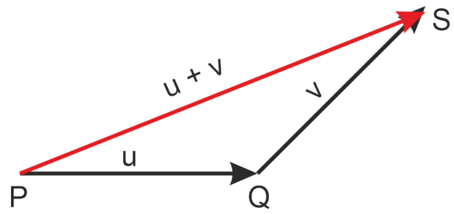


Remark: Following the paper and drawing shapes analogy in the beginning, note that everything we draw and interpret on paper, naturally applies to the three-dimensional space. That is, instead of just choosing a point and a line on a 2D paper, think of the line as a rope that goes from one side to the other side of your room. Choosing two points on the rope would be an equivalent example to the points (A, B) discussed above. From this point beyond, we develop our imaginations in 3D space instead of a 2D plane.

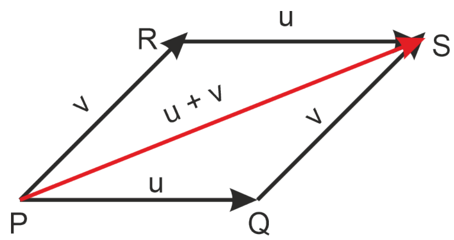
Vectors in 3D space: Mathematical Properties

Generally speaking, vectors can be added together, subtracted from one another, and be multiplied by a scalar. These properties are really nothing but the consequences of our geometric interpretations, if we hold on to the definitions we have developed so far.

Addition:



For instance, if the two non-parallel vectors u and v are represented by directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{QS}$, then the addition $u+v$ can geometrically mean that we start from the point P, travel all the way to the point Q, and then again travel from Q all the way to S. That is, we have actually traveled from point P to the point S (see figure above).
Note the fact that u and v are a set of directed line segments, meaning that we could move them around in 3D space, without any loss of generality. Therefore, if we set them to emerge from a common start point, it is possible to span a parallelogram by the two vectors. The vector w which is the result of adding the vectors u and v, is in fact the vector associated to the directed diagonal of the parallelogram, starting from the common start point of the two vectors.



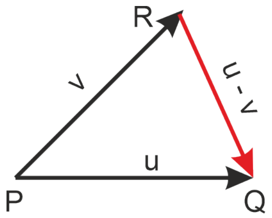
We state that there are further computational rules for vector addition that can easily be verified with geometrical interpretations. Supposing that u, v, and w are three vectors in space:
Commutativity law: $u + v = v + u$
Associativity law: $u + (v + w) = (u + v) + w$
Cancellation law: If $u + v = u + w$ then $v = w$
Neutral element law: $u + 0 = u$

Subtraction:

Vector subtraction is directly defined by what we understand about addition. Imagine a vector v in space. The vector $(-v)$ has exactly the same magnitude as v, but only points in the opposite direction. With that being said, subtracting the two vectors u and v could be in fact interpreted as adding the vector u to the vector $(-v)$. We therefore write:

$$u–v = u + (-v)$$

Geometrically speaking, the vector $w = u - v$ is illustrated as the other directed diagonal of the parallelogram spanned by u and v, which has its starting point on the end of v pointing to the end of u.



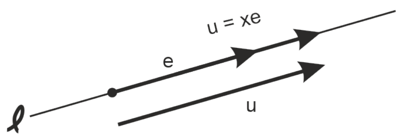
Multiplication by a scalar:

Perhaps the most useful property in understanding how the geometrical aspects of the world we know are dominated by vectors, is by studying the fact that vectors can be multiplied by a scalar (i.e., number). If s is a real number and u a vector in 3D space, then su is in fact, a *rescaled version* of u, and if we denote the magnitude of u by $||u||$, then the vector su has the magnitude $s||u||$.
At this point, this may seem not an important aspect to consider, but the essential role of rescaled versions of vectors comes to attention when we discuss the so-called *Basis*. We can therefore conclude that if two non-zero vectors are parallel, one can be written as the scalar multiple of the other.

Basis: The core of the reality we know

1D Space:

Let's get back to where we started: A line.
Assuming that we have a line called ℓ in space, and furthermore, we have a vector u that is parallel to this line. Now, if we choose a vector on the line ℓ, name it e, the vectors u and e should also be parallel (Since u is parallel to ℓ and e is on ℓ.) Recall that u is a set of directed line segments and therefore could be moved in space. Assuming that we move the vector u and bring it on the line ℓ, It would be visually clear that u is a scalar multiple of e. That is, a number x times vector e, gives us the vector u. And therefore, since u can be represented by a directed line segment on ℓ, we say that "u belongs to ℓ"



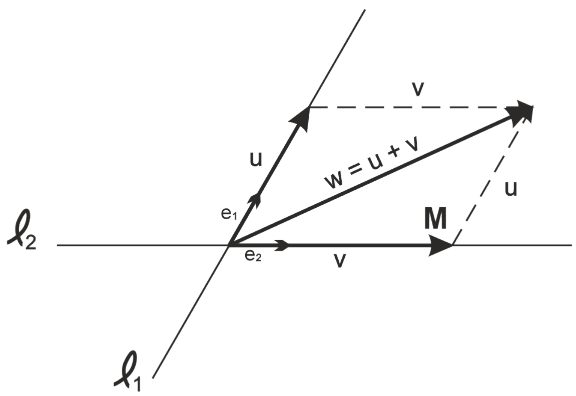
How can we generalize this idea? Well, for any other vector u which belongs to ℓ, there exists a unique real number x. Such that:

$$u = xe$$

And just like that, the vector e becomes a so-called *Basis* for the line ℓ and x becomes the *Coordinate* of u with respect to the basis e.

2D Space:

Now, suppose we have two non-parallel lines ℓ_1 and ℓ_2 which intersect one another at a point. Judging by what we have already learned about lines in space, we know that for every vector u which belongs to ℓ_1, there exists a unique real number x_1. Such that: $u = x_1 e_1$. The very same properties and interpretations apply to ℓ_2: for every vector v which belongs to ℓ_2, there exists a unique real number x_2. Such that: $v = x_2 e_2$. Since ℓ_1 and ℓ_2 intersect each other at a point, a parallelogram could be spanned by their vectors u and v:



Adding the two vectors u and v would give us the diagonal for the spanned parallelogram. The vector $w = u + v$ would therefore become in the form:

$$w = x_1 e_1 + x_2 e_2$$

Just like how it was observed for a one-dimensional figure such as a line, we now observe that for different coordinates x_1 and x_2, the vector w would be inside the plane M spanned by the two vectors u and v. In more general terms we can say that a vector w belongs to a plane M, if w can be represented by a line segment in M. On the other hand, every line segment on M could be decomposed into a sum of the vector $u = x_1 e_1$ and $v = x_2 e_2$ for some real numbers x_1 and x_2. It is therefore clear to conclude that the two non-parallel vectors e_1 and e_2 are the so-called *Basis* of the plane M, and x_1 and x_2 are the coordinates of the vector w with respect to the basis e_1, e_2.

3D Space:

Finally, we arrive at what we could perhaps intuitively grasp better. We take what we learned about vectors, lines and planes in two-dimensional space and try to seek a meaning for them in the 3D space. By reviewing what we have developed so far for one and two-dimensional spaces, it becomes somehow predictable that a 3D space brings an e_3 in play, just as how a 2D space required an e_2 to build up its structure. Here, we are not only trying to mathematically describe all vectors in just a plane, but we're seeking a general definition for all vectors in space. To do so, we go down the same path as we did for the former two cases. Assuming that we have three vectors e_1, e_2, e_3 which do not lie on the same plane, then every vector u can be written as

$$u = x_1e_1 + x_2e_2 + x_3e_3.$$

How can we visualize this? How is this even true? The core of our visualization would once again be a very simple geometrical figure: The line. Imagine three lines ℓ_1, ℓ_2, ℓ_3, in space. ℓ_2 and ℓ_3 intersect at a point and a plane M is spanned by the two vectors u_2 and u_3 on the lines ℓ_2 and ℓ_3 respectively. The line ℓ_1 which does not lie on M, intersects the plane M at a point.
As we already observed in the first case, the vector u_1 belongs to the line ℓ_1 only if $u_1 = xe_1$, with e_1 being the basis of ℓ_1. Naturally, the same interpretation applies to u_2 and u_3.
Now if u is a random vector in space (illustrated in pink in the following figure), it can be decomposed into a sum of three vectors u_1, u_2, and u_3 such that:

$$u = u_1 + u_2 + u_3 = x_1e_1 + x_2e_2 + x_3e_3$$

In simpler terms, the addition of two vectors u_2 and u_3 would naturally provide us with a vector inside M. Now, by adding this vector to u_1, the result vector would be outside of that plane, somewhere in space with respect to the basis e_1, e_2, and e_3.
x_1, x_2, and x_3 are called the *Coordinates* of the vector u and could be abbreviated as $u = (x_1, x_2, x_3)$
As a final mathematical terminology, we say that the vectors e_1, e_2 and e_3 are *Linearly Independent* and consider the vector u as a *Linear Combination* of these vectors.